Abstract

In this thesis, we consider the state estimation problem of constrained discrete-time dynamical systems. More specifically, we develop and analyze moving horizon estimation (MHE) schemes and algorithms within the novel framework of *proximity moving horizon estimation*. In particular, we address the theoretical analysis of the corresponding MHE formulations and develop numerically efficient algorithms for the online implementation.

The first main goal of this thesis is to provide constructive MHE design procedures with desirable guarantees such as nominal stability and robustness of the underlying estimation error. These design procedures are tailored to the considered class of dynamical systems, covering time-invariant and time-varying linear and nonlinear systems, and the guarantees do not require any assumption on the estimation horizon length. In addition to the theoretical results, we study the design and tuning of the performance criteria in proximity MHE from a probabilistic perspective and demonstrate how to ensure a satisfactory performance given prior knowledge on the system disturbances.

The second main goal of this thesis is to develop computationally efficient and reliable MHE algorithms that do not require the exact solution of the constrained MHE problem at each time instant, but rather yield suboptimal state estimates after a limited number of iterations. By taking the dynamics of the employed optimization algorithm into account in the theoretical analysis, we derive sufficient conditions under which desirable stability and performance properties of the estimator hold for any number of optimization algorithm iterations, including the case of a single iteration per time instant. This leads to a novel class of *anytime MHE algorithms* where rigorous guarantees are ensured independently of the number of internal iterations, allowing for a trade-off between computational effort and accuracy of the state estimate.

The obtained theoretical results and practical benefits of the proposed proximity MHE approaches are demonstrated with different numerical examples from the literature.

Deutsche Kurzfassung

Diese Arbeit befasst sich mit der Zustandsschätzung von zeitdiskreten dynamischen Systemen mit Beschränkungen. Insbesondere werden Verfahren und Algorithmen der Zustandsschätzung mit bewegtem Horizont (engl. Moving Horizon Estimation, MHE) innerhalb des neuartigen Grundgerüsts der *Proximity Moving Horizon Estimation* entwickelt und analysiert. Dazu werden theoretische Analysen der entsprechenden MHE-Formulierungen eingesetzt und numerisch effiziente Algorithmen für die Online-Implementierung entwickelt.
Das erste Hauptziel dieser Arbeit besteht darin, konstruktive MHE-Entwurfsverfahren mit relevanten Eigenschaften wie nomineller Stabilität und Robustheit des Schätzfehlers bereitzustellen. Diese Entwurfsverfahren sind speziell auf die betrachtete Klasse dynamischer Systeme, welche zeitinvariante und zeitvariante lineare und nichtlineare Systeme abdeckt, zugeschnitten. Die sich ergebenden Stabilitäts- und Robustheitsgarantien erfordern keine Annahmen über die Länge des zurückliegenden Messhorizonts. Zusätzlich zu den theoretischen Ergebnissen wird der Entwurf und die Feinabstimmung des Gütekriteriums der Proximity-MHE aus einer probabilistischen Perspektive untersucht. Damit wird gezeigt, wie zufriedenstellende Schätzergebnisse bei Vorwissen über die Systemstörungen sichergestellt werden können.
Das zweite Hauptziel dieser Arbeit ist die Entwicklung rechentechnisch effizienter und zuverlässiger MHE-Algorithmen, die keine exakte Lösung des beschränkten MHE-Problems in jedem Zeitschritt erfordern, sondern eine suboptimale Schätzung der Systemzustände nach einer begrenzten Anzahl von Iterationen liefern. Indem die Dynamik des verwendeten Optimierungsalgorithmus in der theoretischen Analyse berücksichtigt wird, lassen sich hinreichende Bedingungen ableiten, unter denen wünschenswerte Stabilitäts- und Performance-Eigenschaften des Schätzers für eine beliebige Anzahl an Iterationen des Optimierungsalgorithmus gelten. Dies schließt den Fall einer einzigen Iteration pro Zeitschritt ein. Damit ergibt sich eine neuartige Klasse von *Anytime-MHE-Algorithmen*, bei denen rigorose Garantien unabhängig von der Anzahl an internen Iterationen gewährleistet sind, was eine Abwägung zwischen Rechenaufwand und Genauigkeit der Zustandsschätzung ermöglicht.
Die erzielten theoretischen Ergebnisse und praktischen Vorteile der vorliegenden Proximity-MHE-Ansätze werden anhand verschiedener numerischer Beispiele aus der Literatur demonstriert.

Chapter 1

Introduction

1.1 Motivation

State estimation is the process of reconstructing an unmeasurable state of a dynamical system from a given model in combination with the available input and measurement data. State estimation is of paramount importance in many engineering domains such as signal processing, geodesy, astronomy, communication, control, and learning, where the success of many of the underlying applications directly relies on the quality of the estimated states. The problem of interpreting observations and computing estimates and predictions is not of recent origin, and the early history dates back to Galileo's statistical analysis of astronomical observations in 1632 (Kailath (1974)). Gauss apparently first discovered the least-squares method in 1795, though it was officially independently introduced and published by Legendre in 1805 (Sorenson (1970)). In his book on planetary orbits published in 1809, Gauss discussed the estimation of the six parameters that determine an elliptical orbit on the basis of more than six observations (Sprott (1978)). The invention of least-squares estimation has provided the foundation for many estimation theories and techniques in the 19th and 20th centuries (Sorenson (1970)). One of the most widely-used state estimators which had a huge impact on the field of state estimation and forms the backbone of numerous applications is the Kalman filter. In his pioneering work, Kalman (1960b) provided a computationally attractive recursive solution to the minimum variance estimation problem for linear systems subject to Gaussian disturbances. Since then, many extensions and generalizations of the Kalman filter were developed, which is revealed by the existing vast and rich literature (see, e.g., Särkkä (2013)). In addition to the fact that it allows for a fast implementation in real-time, the linear Kalman filter exhibits important theoretical guarantees such as stability of the estimation error under standard assumptions (Jazwinski (1970)). However, these guarantees as well as a good practical performance of the Kalman filter are not ensured when nonlinear systems are considered or Gaussian assumptions are violated. Moreover, prior knowledge on the system in form of physical state constraints cannot be directly incorporated into the estimation process.

One successful approach that overcomes the aforementioned issues is moving horizon estimation (MHE). MHE is an optimization-based approach that estimates the state of a dynamical system by solving a suitable optimization problem at each time instant. Hereby, MHE takes a fixed and limited number of the most recent input and measurement data into account and the considered horizon of data is moved forward in time (in a receding horizon manner) when a new measurement becomes available. The main benefits of MHE are (i) its capability to exploit physical constraints on states and disturbances by including

them directly in the underlying optimization problem, which may lead to an improved performance compared to the Kalman filter (Muske et al. (1993)), (ii) the handling of general nonlinear systems, (iii) the ability to consider suitable performance criteria designed specifically for the problem at hand, and (iv) tractable online computational complexity compared to the full information estimation problem (Rawlings et al. (2017)). Since the original idea of MHE was presented by Jazwinski (1968) and stability properties were first explored by Ling and Lim (1999); Michalska and Mayne (1993, 1995), MHE has gathered more attention during the last decades from both theoretical and practical viewpoints. In fact, there is a growing number of application examples of MHE, such as automotive applications (Andersson and Thiringer (2018)), mobile and humanoid robots (Bae and Oh (2017); Liu et al. (2016)), state of charge estimation (Hu et al. (2017); Shen et al. (2016)), spacecraft attitude estimation (Huang et al. (2017); Qin and Chen (2013); Vandersteen et al. (2013)), water level regulation in inland navigation (Segovia et al. (2019)), and optimal dosing of cancer therapy (Chen et al. (2012)). Moreover, design methods of MHE have been proposed in the literature which take into account large scale systems (Farina et al. (2010); Haber and Verhaegen (2013); Schneider et al. (2015); Zhang and Liu (2013)), hybrid systems (Ferrari-Trecate et al. (2002)), systems with multi-rate measurements (Gu et al. (2019)), and binary sensors (Battistelli et al. (2017)).
Although the extent of the literature validates the benefits of MHE and reports a significant progress on its theoretical analysis, design, and application, we enumerate the following fundamental challenges in the MHE research that might reduce its practical usefulness.

- Optimization algorithm of MHE:
  In many demanding applications such as in the field of automation, a crucial requirement on the estimator is to process and manipulate large amounts of measurement data in a very fast and reliable fashion. However, MHE can be computationally intensive since it involves the online solution of an optimization problem each time a new measurement becomes available. Thus, an efficient and real-time capable implementation of MHE with theoretical guarantees is required for practical and safety critical applications which fulfills the constraints imposed by the limited computational resources. By contrast, most of the MHE approaches with stability guarantees do not consider the numerical implementation of the underlying optimization problem in the theoretical analysis. More specifically, stability or convergence results are usually presented with respect to the exact solution of the constrained optimization problem and do not take into account the internal iterations of the employed optimization algorithm. Moreover, there is a lack of theoretical results that demonstrate how the accuracy of the suboptimal estimate, which can be characterized by the number of executed optimization algorithm iterations, affects the performance of MHE algorithms. Although these issues have drawn more attention in the recent years (Alessandri and Gaggero (2020); Findeisen et al. (2018); Kong and Sukkarieh (2018); Schiller et al. (2020); Zou et al. (2020)), theoretical studies that address both stability as well as performance of MHE algorithms under rather mild assumptions are rare in the literature and still require investigation.

- Stability theory of MHE:
  As already mentioned, a reliable state estimation strategy with important theoretic properties such as stability of the estimation error is viable for the success of the considered application. Especially for nonlinear systems, the problem of the synthesis

of a state estimator for stability purposes is significant and difficult. This challenge has been addressed in the literature and numerous contributions have been made for the development of MHE design methods in order to guarantee stability of the underlying estimation error. In a notable and rather standard MHE approach (Rawlings et al. (2017)), this problem is tackled by providing a sufficient condition on the approximation of the so-called arrival cost, which summarizes the past history of data not included in the MHE problem. However, the derived condition is rather restrictive and hard to meet for a general nonlinear system. Except for the special case of linear systems, quadratic cost, and convex constraints, developing constructive methods for its satisfaction remains a key issue as pointed out by Rawlings et al. (2017). Nevertheless, there exist alternative MHE approaches which ensure stability and robustness even in the nonlinear case thanks to offline design procedures, see among others the interesting works of Alessandri et al. (2010); Müller (2017); Sui and Johansen (2014) and the detailed discussions in Chapter 2. However, many of the established results either rely on a sufficiently long estimation horizon length, or hold due to sufficient conditions which are limited to a quadratic cost and explicitly depend on global properties of the considered dynamical system, which are often hard to verify in the general nonlinear case. Hence, there is a need for conceptually simple and constructive design procedures of nonlinear and linear MHE schemes which ensure desirable theoretical guarantees.

- Robustness of MHE with respect to disturbances:
  The ability to cope with various disturbances such as measurement outliers, which occur for instance due to sensor malfunction or failures of data transmitters, can be a crucial requirement in many applications (Alessandri and Awawdeh (2016); Ben-Gal (2005); Hawkins (1980); Rousseeuw and Leroy (2005)). Hence, it is desirable to include in the MHE problem meaningful stage costs that describe statistical characteristics of the system disturbances in order to accurately estimate the state of the underlying system. For example, if it is known that outliers occur in the measurements, instead of penalizing the output residuals using the standard ℓ_2-norm, the so-called Huber penalty function can be used as stage cost. This penalty is quadratic for small residuals but linear for large residuals, which makes it less sensitive to outliers. Furthermore, the sparsity promoting ℓ_1-norm can be employed, which has been highly successful in many signal processing applications (Candes et al. (2008); Donoho (2006)). However, there has been little discussion about MHE approaches with stability guarantees in which the commonly-used least-squares formulation is not adapted. This issue is also highlighted in several recent developments (Chu et al. (2012); Geebelen et al. (2013); Haverbeke (2011); Kouzoupis et al. (2016)), which propose to use the ℓ_1-norm or the Huber penalty function instead of the traditional ℓ_2-norm in MHE in order to account for outliers. Despite the fact that these estimators exhibit improved performance, the stability properties of the underlying estimation error are not investigated. Hence, it is desirable to develop MHE approaches that allow for flexible performance criteria and for which theoretical guarantees still hold.

These challenges create an incentive for developing MHE procedures which are easy to implement, easy to design, and with system theoretic guarantees that hold for a broad variety of system classes and performance criteria. More specifically, the goal of this thesis is to provide a unified framework for the design, analysis and implementation of moving horizon estimators, which are theoretically sound with desirable guarantees, computationally efficient, and allow for a flexible formulation of the underlying optimization problem.

1.2 Contributions and outline of the thesis

In this thesis, we introduce the novel framework of *proximity moving horizon estimation* of constrained discrete-time linear and nonlinear systems. It is based on the general conceptual idea of employing a stabilizing analytical a priori solution and combining it with an online optimization in order to obtain an improved performance without jeopardizing the stability properties. Based thereon, we devote our attention to the theoretical analysis of MHE as well as to the practical challenge of computational efficiency. More specifically, we first address MHE theory with a focus on MHE *schemes*, where stability is investigated under the assumption that a solution of the optimization problem is available at each time instant. The underlying cost function consists of two parts: the first part includes a general, possibly nonsmooth convex stage cost which can handle outliers and other data specific characteristics; the second part is a suitable proximity measure to a stabilizing a priori estimate, from which stability can be explicitly inherited. Second, we address MHE implementation with a focus on MHE *algorithms* or *iteration schemes*, where the dynamics of the optimization algorithm are taken into account in the stability analysis. The underlying optimization algorithm computes the next iterate as the solution of a simple convex optimization problem, where a first-order approximation of the MHE stage cost and a suitable proximity measure to the previous iterate are minimized. The optimization algorithm terminates after a limited number of iterations and is warm-started by a stabilizing a priori estimate, from which stability can be implicitly inherited.
In the proposed MHE approaches, the proximity term is characterized by the so-called Bregman distance, which constitutes a measure of distance between two points and is constructed in terms of a strictly convex function (Censor and Zenios (1992)). Moreover, the a priori estimate is generated from a model-based and convergent recursive estimator for which a suitable Lyapunov function of the error dynamics is known. While the stability components consisting of the Bregman distance and the a priori estimate can be designed *offline*, the performance of the estimator is specified through the stage cost and the most recent batch of input and measurement data in the *online* optimization. A graphical illustration of this concept is presented in Figure 1.1.

The methodological core for the design and analysis within the proximity MHE framework is from the field of state estimation and the powerful class of proximal methods for solving nonsmooth convex optimization problems (Beck and Teboulle (2003); Güler (1991); Parikh and Boyd (2014); Rockafellar (1976); Teboulle (1992)). The theory therein is very rich and serves as an efficient mathematical tool for investigating important theoretic properties of the developed proximity MHE approaches. Given the central role played by proximal methods in deriving many of the results presented in this thesis, and since the proposed approaches use the Bregman distance as a proximity measure through which stability can be explicitly or implicitly inherited from the stabilizing a priori estimate, we refer to our framework as "proximity" MHE. Based on this discussion, we summarize in the following the main contributions and present in more detail the outline of this thesis.

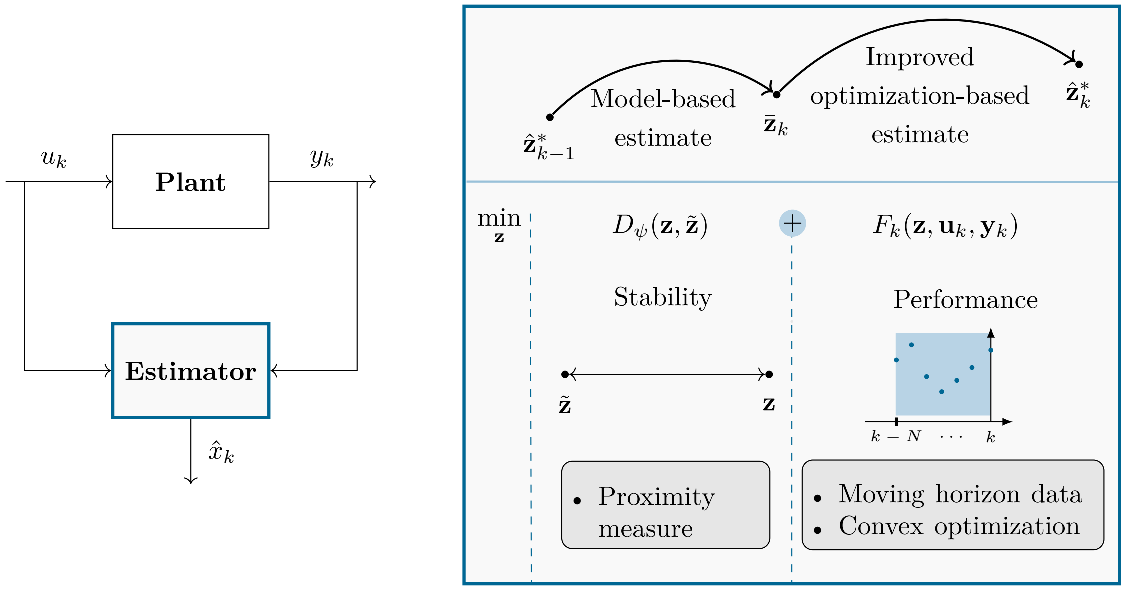

Figure 1.1: Proximity moving horizon estimation: At each time instant k, a stabilizing a priori estimate $\bar{\mathbf{z}}_k$ is computed by updating (predicting) the previous MHE solution $\hat{\mathbf{z}}^*_{k-1}$ through a model-based analytical estimator. Given the calculated a priori estimate, an improved optimization-based estimate $\hat{\mathbf{z}}^*_k$ based on which $\hat{x}_k$ is generated is obtained by solving the proximity MHE problem. In the associated optimization problem, a suitable Bregman distance D_ψ is used as proximity measure to $\tilde{\mathbf{z}}$ and combined in an online optimization with the performance criterion $F_k(\mathbf{z}, \mathbf{u}_k, \mathbf{y}_k)$, which is based on the horizon of most recent input and measurement data $\mathbf{u}_k, \mathbf{y}_k$. In chapter 3, where a solution of the proximity-based MHE problem is computed at each time instant, $\tilde{\mathbf{z}}$ is exactly the a priori estimate $\bar{\mathbf{z}}_k$. In chapter 4, where a fixed number of iterations is executed at each time instant, an iterative optimization update is carried out, in which the Bregman distance keeps $\mathbf{z}$ close to the previous iterate $\tilde{\mathbf{z}}$ and $\bar{\mathbf{z}}_k$ is only used as a warm start strategy. In both cases, the combination ensures both stability and performance of MHE.

Main contributions

This thesis consists of two central contributions, each of which is related to the two main chapters of the thesis.

- The first main contribution of the thesis, Chapter 3, is concerned with the design and analysis of proximity MHE *schemes* for discrete-time linear and nonlinear systems subject to convex constraints. In this chapter, we present the novel proximity-based formulation of the MHE optimization problem, where the cost function includes a convex stage cost as well as a Bregman distance centered around a stabilizing a priori estimate. The main objective is to derive sufficient conditions on the Bregman distance and the a priori estimate such that stability properties of the underlying estimation error can be established. Based on these conditions, we propose explicit design procedures which are tailored to the considered system class. A technical contribution is a novel and unified Lyapunov-based approach for the stability analysis, which allows MHE to inherit the stability properties of the employed a priori estimate by choosing the Bregman distance as a Lyapunov function. Furthermore, we embed the proposed MHE formulation into a stochastic framework in order to obtain a Bayesian interpretation, where the stabilizing model-based estimator is considered as a priori knowledge obtained offline, before observed data from the most recent measurements are included. Based thereon, we describe how the design and tuning of the stage cost can be guided by the statistics of the disturbances.

- The second main contribution of the thesis, Chapter 4, is concerned with the design and analysis of proximity MHE *algorithms* for discrete-time linear and nonlinear systems subject to convex constraints. In this chapter, we present a novel proximity-based MHE iteration scheme, which reduces computational burden by performing only a limited number of optimization algorithm iterations each time a new measurement is received. More specifically, a suitable gradient-based proximal algorithm for minimizing the sum of convex stage cost is initialized based on a stabilizing a priori estimate and used to deliver a suboptimal estimate after single or multiple iterations per time instant. By means of a rigorous Lyapunov analysis, we derive conditions under which stability of the underlying estimation errors is ensured for any arbitrary number of optimization algorithm iterations. Thereby, we obtain a so-called *anytime MHE algorithm*, where stability is guaranteed independently of the executed number of internal iterations, and which allows the user to achieve a trade-off between computational complexity and accuracy of the state estimate. Although this anytime property is induced from the stabilizing a priori estimate, the (suboptimal) bias of the employed recursive estimator used to construct the a priori estimate is fading away with each iteration since it is only used to warm start the optimization algorithm. This is an implicit stabilizing regularization approach of the a priori estimate, which is conceptually different from the explicit stabilizing regularization used in Chapter 3 to ensure stability. Furthermore, we study the performance of the proximity MHE algorithm in terms of a regret analysis, which is widely used in the field of online convex optimization to characterize performance. By adapting this notion of regret to our setting, we provide performance guarantees in terms of rigorously derived regret bounds. The established bounds allow to measure the real-time regret of the proposed algorithm that carries out only finitely many optimization iterations (due to limited computing power and/or minimum required sampling rate) relative to a comparator algorithm that gets instantaneously an optimal solution from some oracle.

Thesis outline

Chapter 2: Background. In this chapter, we present the basic principle of moving horizon estimation and a short review on some of the relevant and related literature, which serves as a basis for comparison with the results derived in this thesis. Moreover, we provide a brief overview on proximal methods for convex optimization.

Chapter 3: Proximity-based MHE schemes In this chapter, we first start by presenting the problem setup and formulating the proximity-based MHE optimization problem. Second, we focus on the class of linear time-varying systems, for which we establish global uniform exponential stability of the estimation error in the absence of

disturbances, as well as input-to-state stability when additive process and measurement disturbances affect the system. A Bayesian interpretation of the proximity MHE scheme is also established and the relationship to Kalman filtering is investigated. For the special class of linear time-invariant systems, we show that the theoretical results hold under very mild assumptions, where only detectability of the system matrices is needed. Third, we extend our results to nonlinear systems and establish local uniform exponential stability of the underlying estimation error under suitable assumptions. Moreover, we consider a class of nonlinear systems that can be transformed into systems that are affine in the unmeasured state. For these systems, we formulate the proximity MHE problem as a convex optimization problem and show that the estimation error is globally uniformly exponentially stable. Due to the simple proximity-based design, the theoretical guarantees derived in this chapter hold for any horizon length and irrespectively of the convex stage cost being used. Finally, we illustrate the benefits of the proposed MHE approaches using numerical examples from the literature.
The results of this chapter are based on (Gharbi and Ebenbauer (2018, 2019a,b, 2020); Gharbi et al. (2020a)).

Chapter 4: Anytime proximity-based MHE algorithms In this chapter, we first present the estimation problem of interest. Second, we describe the proximity-based MHE iteration scheme in detail. At each time instant, the underlying optimization algorithm is warm-started by a stabilizing a priori estimate and delivers a state estimate in real-time. Third, we consider linear systems and prove that global uniform exponential stability of the resulting estimation error can be ensured for any number of optimization algorithm iterations. In addition, we establish performance guarantees of the proposed MHE iteration scheme in terms of regret upper bounds. Our results show that both exponential stability and a sublinear regret can be guaranteed, where the latter can be rendered smaller by increasing the number of optimization iterations. Fourth, we show that the stability results derived in the linear case can be extended to nonlinear systems by proving local uniform exponential stability of the underlying estimation error. Thanks to the simple proximity-based design of the MHE algorithm, the stability and performance guarantees derived in this chapter hold for any number of optimization algorithm iterations, any horizon length, and for a rather general convex stage cost that is not necessarily quadratic. Finally, we use numerical examples to illustrate the obtained stability and regret results as well as the computational efficiency of the proposed iteration scheme.
The results of this chapter are based on (Gharbi and Ebenbauer (2021); Gharbi et al. (2020b, 2021)).

Chapter 6: Conclusions. In this chapter, we summarize the main results of this thesis and provide perspectives for future research.

Appendices. In Appendix A, we provide stability definitions of discrete-time systems which are employed throughout this thesis. Moreover, we recall how stability properties can be established using Lyapunov functions.

Chapter 2

Background

In this chapter, we give a brief review on optimization-based state estimation by introducing full information estimation (FIE) and moving horizon estimation (MHE) and presenting some approaches from the literature that guarantee stability of the estimators. Furthermore, we provide a short background on proximal methods for solving convex optimization problems which serves as a starting point towards the proximity-based formulation and analysis of MHE in the subsequent chapters of the thesis.

2.1 Moving horizon estimation

In this section, we briefly present standard stability results for FIE and MHE in a deterministic setting, where the considered system class as well as the main assumptions are based on (Rawlings et al., 2017, Chapter 4).
Consider a discrete-time nonlinear system of the form

$$x_{k+1} = f(x_k, w_k), \tag{2.1a}$$
$$y_k = h(x_k) + v_k, \tag{2.1b}$$

where $k \in \mathbb{N}$ denotes the discrete time instant, $x_k \in \mathbb{R}^n$ the state vector, $y_k \in \mathbb{R}^p$ the measurement vector, and $w_k \in \mathbb{R}^{m_\mathrm{w}}$ and $v_k \in \mathbb{R}^p$ account for unknown process and measurement disturbances, respectively. Moreover, the initial condition $x_0 \in \mathbb{R}^n$ of system (2.1) is unknown. The functions $f : \mathbb{R}^n \times \mathbb{R}^{m_\mathrm{w}} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are assumed to be continuous. The state and disturbances are known to verify the following constraints

$$x_k \in \mathcal{X}_k \subseteq \mathbb{R}^n, \quad w_k \in \mathcal{W}_k \subseteq \mathbb{R}^{m_\mathrm{w}}, \quad v_k \in \mathcal{V}_k \subseteq \mathbb{R}^p, \qquad k \in \mathbb{N}, \tag{2.2}$$

where the sets $\mathcal{X}_k$, $\mathcal{W}_k$, and $\mathcal{V}_k$ are nonempty and closed, with $0 \in \mathcal{W}_k$, and $0 \in \mathcal{V}_k$. We let $x\,(k; x_0, \mathbf{w}_k)$ refer to the solution of system (2.1) at time k with initial state x_0 and disturbances $\mathbf{w}_k = \{w_0, \cdots, w_{k-1}\}$. Note that for simplicity of presentation, and analogous to the problem setup in (Rawlings et al., 2017, Chapter 4), inputs u_k are not considered in system (2.1). Nevertheless, the subsequent assumptions and results can be extended to systems of the form $x_{k+1} = f(x_k, u_k, w_k)$ without any particular conceptual difficulties.
The goal is to find at each time instant k an estimate $\hat{x}_k$ of the state x_k given the model (2.1), the constraints (2.2), and the available measurements $\{y_0, y_1, \ldots, y_{k-1}\}$. In FIE, the state estimation problem is tackled by solving at each time instant k a suitable constrained optimization problem that incorporates all the available past measurements.