1 Introduction

1.1 Motivation

With respect to the future urban mobility, modern electrical bicycles, advanced motorcycles and innovative two-wheeled vehicles are arresting enormous amount of attention. The dynamic behaviour of two-wheeled vehicles, especially their self-stability, has been matter of scientific studies for many decades. In fact, as stated in [MP11], the self-stability of bicycles as dynamical systems have been studied as early as in 1910. The necessity of serious investigations of two-wheeled vehicles from a control engineering perspective, however, has increased in the last several years. Comparing to four-wheeled vehicles, an autonomous driving bicycle for passenger transportations may today sound like science-fiction or even unnecessary. Yet, current evolution of the technology towards intelligent, automated and highly connected mobility of the future requires the machine intelligence to be able to handle two-wheeled vehicles as well.

The advantages of two-wheeled vehicles with respect to energy consumption, parking space and environmental friendliness are self-evident. An autonomous two-wheeled vehicle is for instance, among many other conceivable application fields, a noticeable candidate for future automated delivery systems in urban environments. Therefore, model-based control and optimal trajectory planning for such vehicles will remain inevitable matter of research and development in the future. The results are currently, and will in short term be further used for developing safety-increasing assistant systems for existing vehicles, e.g. [KMT09]. In long term, developing innovative autonomous two-wheeled vehicles[1] is likely to require even more intensive research, as it is the case for autonomous driving cars today.

While a large portion of the available know-how on sensor data processing, localisation methods and similar technologies are transferable from cars, due to the special dynamical behaviour of two-wheeled vehicles, methods for trajectory planning and tracking control are not trivially reusable. Therefore, a reliable and yet usable vehicle model as well as a systematic approach to motion control for two-wheeled vehicles are essential, to which the present work makes a contribution.

[1]For instance C1 from Lit Motors: `www.litmotors.com/product`

1.2 Objective and contribution of this work

The available literature contains, on the one hand, extensive studies of the dynamics of bicycles based on quantitative considerations and numeric calculations, which lead to sophisticated models that are not trivially usable for systematic control synthesis. On the other hand, simpler bicycle models are used for purposes of trajectory planing and motion control synthesis. These models however, have either a constrained range of validity, such as models for constant velocities, or, ignore parts of bicycle's physics to keep the resulting equations simple. The main objective of this thesis is to fill this gap and to present a unifying approach to modelling and control for autonomous two-wheeled vehicles. The resulting model shall be generally valid and physically detailed enough to represent the characteristic dynamical behaviour, and at the same time, be proper for a systematic control synthesis. Furthermore, an extension of the model by a rider or further actuators is required to be possible in a systematic way.

To this end, the Hamiltonian framework is chosen. The main contribution of this work is to propose a vehicle model as a port-Hamiltonian system, which is derived using an automatable scheme. The model represents the bicycle's physics such as the self-stability and is, at the same time, directly usable for model-based trajectory planing as well as for passivity-based trajectory tracking controller design. The proposed methodic approach to model derivation is a general procedure that can be extended in a systematic way.

Furthermore, a trajectory tracking controller is designed that is physically interpretable, is valid for the sophisticated vehicle model, and, also robust against parameter uncertainties. To this end, existing approaches of passivity-based controller design are extended and adjusted for two-wheeled vehicles.

The outline of this thesis and particular contributions in every chapter are given below.

Chapter 3
In this chapter, the physics of bicycles is summarised and facts about their dynamics are explained, which are recalled throughout the entire work. Furthermore, the so-called linear benchmark model is presented which is taken as a basis to validate dynamical behaviour of the vehicle model proposed in this thesis.

Chapter 4
In this chapter, the main contribution of this work is presented. That is proposing an approach to derive a physically realistic, yet usable model for two-wheeled vehicles for control synthesis, as well as for trajectory planning. Some geometric considerations at the beginning of the model derivation make sure that physical phenomenon such as self-stability is represented by the resulting model, that is validated later in the chapter. The model is validated by comparison to the benchmark model from the literature, as well as by a series of simulation scenarios, which are designed to demonstrate different aspects of

the dynamical behaviour of the vehicle. Further, a methodical approach is proposed for structure preserving simplification of the equations of motion as well as extension by a new rigid body, for instance an active rider. A part of the content of this chapter is published in [TL18b] and [TL18a].

Chapter 5
In this chapter, optimal trajectory planning methods are used to demonstrate the usability of the proposed model as well as the introduced simplification. It is shown that using the systematic model simplification, trajectories can be planned for the vehicle with a lower computation effort without significant loss of model accuracy. A part of the content of this chapter is published in [TL19].

Chapter 6
In this chapter, a passivity-based trajectory tracking controller is developed using the proposed vehicle model. To this end, existing approaches are extended and various simulations are run to demonstrate the performance of the closed loop under different conditions. Some initial ideas used in this chapter were published in [TO17]. A part of the content of this chapter is, furthermore, published in [TSL18] and [TL18a].

Chapter 7
In this chapter, a prototype two-wheeled vehicle is briefly introduced, which was developed related to this work.

Chapter 8
In this chapter experimental results are presented which demonstrate the functionality of the prototype vehicle and the concept of motion control for two-wheeled vehicles. Furthermore, experimental results are presented for the validation of the proposed model, as well as for the demonstration of its advantage comparing to a widely used nonlinear model from the literature. A part of the content of this chapter is published in [TL19].

1.3 Published content

Related to this work, following papers were published:

[TO17] A. Turnwald and T. Oehlschlägel. Passivity-based control of a cryogenic upper stage to minimize fuel sloshing. *Journal of Guidance, Control, and Dynamics*, 2017.

[TL18b] A. Turnwald and S. Liu. A nonlinear bike model for purposes of controller and observer design. *IFAC-PapersOnLine*, 2018. 9th Vienna International Conference on Mathematical Modelling.

[TL18a] A. Turnwald and S. Liu. Adaptive trajectory tracking for a planar two-wheeled vehicle with positive trail. *In 2018 IEEE Conference on Control Technology and Applications (CCTA)*, 2018.

[TSL18] A. Turnwald, M. Schäfer, and S. Liu. Passivity-based trajectory tracking control for an autonomous bicycle. *In IECON 2018 - 44th Annual Conference of the IEEE Industrial Electronics Society*, 2018.

[TL19] A. Turnwald and S. Liu. Motion planning and experimental validation for an autonomous bicycle. *In IECON 2019 - 45th Annual Conference of the IEEE Industrial Electronics Society*, 2019.

Furthermore, some of the supervised masters theses are listed as:

Adalbert, M. (2016)	Control of port-Hamiltonian systems via generalised canonical transformations
Matheis, N. (2016)	Implementation of a passivity-based controller for an inverted rotational pendulum
Garcia, F. J. R. (2017)	Development and implementation of pose estimation and localization for an autonomous bicycle
Schäfer, M. (2018)	Passivity-based trajectory tracking control for a two-wheeled vehicle
Ahmed, R. (2019)	Model validation for a two-wheeled vehicle using multibody simulation and experimental data
Mouaffo, U. (2019)	Optimal path and trajectory planning for a two-wheeled vehicle using nonlinear dynamics
Muniappan, K. (2019)	Implementation and experimental validation of linear controllers for an autonomous bicycle
Thirumurugan, D. (2019)	Design and experimental investigation of a CMG-based stabilisation of a two-wheeled vehicle

2 Preliminaries

This chapter summarises some of the required definitions and the notation used in this work. Note that detailed explanations, derivations or proofs are omitted on purpose, since those are found in the literature cited within the text of this chapter.

In Section 2.1, the tensor notation, that is mainly used in Chapter 4, is briefly described. Further, some necessary preliminaries from the systems theory are mentioned, especially for Chapters 4 and 6. Finally, some basic definitions are given in context of mechanical systems. In Section 2.2, modelling of constrained mechanical systems is addressed and the derivation of some essential equations from the literature is presented.

2.1 General notations and elementary definitions

2.1.1 Tensor notation

For sake of clarity and comprehension, the tensor notation used in this work, mainly in Chapter 4, is outlined briefly. Tensor notation is a mathematical tool from tensor calculus that is often used in different fields of physics, especially relativity theory, since it allows generic handling of multi-dimensional entities. The following introduction to tensor notation is mainly based on [Sus10] and [Bis16].

Within the present work, entities with more than one components, in other words non-scalars, are denoted using matrix notation or tensor notation in an equivalent meaning. Note that in this thesis, the definitions of vectors or tensors are not considered strictly, rather only the notations and mathematical tools from the corresponding calculus are applied for consistent calculations. Matrices and vectors are denoted by bold symbols, for instance the vector

$$\boldsymbol{f} \in \mathbb{R}^3 .$$

Using the the tensor notation, an entity is denoted by indexes as sub- and/or superscripts. For example

$$f_\alpha \text{ with } \alpha = 1,\, 2,\, 3.$$

Note that it is only a matter of convention whether an index is a sub- or a superscript. This means that the same vector can also be denoted as f^α, as long as this *choice* is consistent throughout the entire calculations.

The number of indexes determines the dimension of the entity. For instance the matrix $\boldsymbol{M}$ can be equivalently denoted by

$$\begin{aligned} M_{\alpha\beta} &:= \boldsymbol{M} \in \mathbb{R}^{n\times m} \\ \text{or } M_{\alpha}^{\beta} &:= \boldsymbol{M} \in \mathbb{R}^{n\times m} \\ \text{or } M_{\beta}^{\alpha} &:= \boldsymbol{M} \in \mathbb{R}^{n\times m} \\ \text{or } M^{\alpha\beta} &:= \boldsymbol{M} \in \mathbb{R}^{n\times m}, \end{aligned}$$

with $\alpha = 1, \cdots, n$ and $\beta = 1, \cdots, m$. Note that for every combination (α, β), the above symbols denote the $\alpha\beta$-component of the matrix $\boldsymbol{M}$.

In tensor calculus, components corresponding to the subscripts are referred to as *co-variants* and those corresponding to superscripts as *contra-variant*. This has to do with the way how tensors are transformed with regard to their basis. In Cartesian coordinates, the co- and contra-variants are identical. In this work, the sub- and superscripts are used to consistently denote specific entities such as generalized coordinates and impulses in accordance to the notation from the book [Blo16].

Once the convention is chosen, every denotation must be consistent and the order of the indexes is substantial. For instance, if the notation $M_{\alpha\beta} := \boldsymbol{M}$ is chosen, the transposed matrix is denoted in tensor form as $M_{\beta\alpha} := \boldsymbol{M}^T$.

The most important advantage of the tensor notation, and the main reason for using them in this work, is the fact that entities with a dimension > 2 can be denoted simply and generically. An example is the three-dimensional tensor

$$B_{\alpha\beta}^{b} \qquad \text{with} \qquad \alpha = \beta = 1 - n_r \text{ und } b = 1 - n_s,$$

which will be defined in Section 2.2. Note that this entity cannot be denoted and calculated with using matrix notation only.

Using the tensor notation, the so-called *Einstein's sum convention* holds. That is, when summing over a particular index, where the index is a subscript in one tensor and a superscript in the other, the sum-symbol is omitted. For instance, the scalar product of two vectors $\boldsymbol{f}$ and $\boldsymbol{g}$

$$\boldsymbol{f}^T \boldsymbol{g} = f_1\, g_1 + f_2\, g_2 + \cdots + f_n\, g_n = \sum_{i=1}^{n} f_i\, g_i$$

is given using tensor notation as

$$\boldsymbol{f}^T \boldsymbol{g} = f_i\, g^i = f^i\, g_i = g_i\, f^i = g^i\, f_i$$

depending on the chosen convention. Note that since the product $g^i\, f_i$ is the product of two scalars, namely the ith component of each vector, the order can be changed.

When the sum convention is applied, the corresponding index vanishes in the product result. For instance

$$p_i = M_{ij}\,\dot{q}^j = M_{ik}\,\dot{q}^k \ .$$

Note that the index over which the summation is applied can be arbitrarily exchanged ($j \rightarrow k$) as long as it is the same index for both tensors. In accordance to the most literature involving tensor notation, often indexes are in fact reused in this work, over which the summation applies, to reduce the number of used letters.

Two tensors can only be added if they have the same indexes in the same order, for example

$$f_\alpha^\beta + g_\alpha^\beta = h_\alpha^\beta \quad \text{or} \quad M_{\alpha\beta} + N_{\alpha\beta} = P_{\alpha\beta}.$$

If a tensor is in the denominator of a fraction is chosen to have its indexes as subscript, it may be given in the nominator exchanging the subscript with the superscript, e.g.

$$f_\alpha = \frac{1}{f^\alpha} \quad \text{or} \quad f^\alpha = \frac{1}{f_\alpha}. \tag{2.1}$$

Therefore,

$$f_\alpha^\beta = g_\alpha^\beta + \frac{1}{h_\beta^\alpha} + \frac{m_\alpha}{n_\beta}. \tag{2.2}$$

is a valid equation since the indexes are consistent.

For more details on tensor calculus and the tensor notation, one may refer to the lectures of Prof. Leonard Susskind ([Sus10]) and also in [DP10].

2.1.2 System theory

A general nonlinear time-invariant system $\sum$ is given by

$$\sum : \begin{cases} \dot{\boldsymbol{x}} & = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} & = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}) \end{cases} , \boldsymbol{x} \in \mathcal{X} \ , \boldsymbol{y} \in \mathcal{Y} \ , \boldsymbol{U} \in \mathcal{U} \tag{2.3}$$

where $\boldsymbol{u}(t)$ is the input vector, $\boldsymbol{y}(t)$ the output vector and the manifolds $\mathcal{U}$ and $\mathcal{Y}$ are the corresponding domains. $\boldsymbol{x} = [x_1, x_2, ...x_n]^T$ is the state vector and the manifold $\mathcal{X}$ is the corresponding domain.

Definition 2.1.1.
The system $\sum$ *is called autonomous, if* $\boldsymbol{u}(t) \equiv \boldsymbol{0}$.

Definition 2.1.2. *[Ada15]*
A system is called ***input-affine****, if it is defined as*

$$\sum\nolimits_{aff} : \begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \end{cases} . \tag{2.4}$$

Definition 2.1.3. *[FS01a]*
The system $\sum_{aff}$ *is called* ***distinguishable****, if for a given trajectory* $\boldsymbol{x}_d(t)$ *satisfying* (2.4)

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_d(t_0)\ ,\ \boldsymbol{y} - \boldsymbol{y}_d \equiv \boldsymbol{0}\ \forall t \in [t_0, t_1]\ \Rightarrow\ \boldsymbol{x}(t) = \boldsymbol{x}_d(t)\ , \forall t \in [t_0, t_1] \tag{2.5}$$

holds.

Definition 2.1.4.
The point $\boldsymbol{x}^*$ *is called an* ***equilibrium*** *for the system* $\sum$ *if*

$$\dot{\boldsymbol{x}}^* = \boldsymbol{f}\left(\boldsymbol{x}^*\right) = \boldsymbol{0}\ . \tag{2.6}$$

Definition 2.1.5. *[Ada15]*
The equilibrium $\boldsymbol{x}^*$ *of the autonomous system* $\sum$ *is said to be* ***locally attractive****, if a neighbourhood* $U(\boldsymbol{x}^*) \subseteq \mathcal{X}$ *exists, such that every initial value*

$$\boldsymbol{x_0} \in U\left(\boldsymbol{x}^*\right) \tag{2.7}$$

leads to a trajectory $\boldsymbol{x}(t)$ *converging to the equilibrium for* $t \to \infty$*. If the neighbourhood is the entire space* $U(\boldsymbol{x}^*) = \mathcal{X}$ *the equilibrium is said to be* ***globally attractive****.*

Definition 2.1.6. *[Lun16]*
The equilibrium $\boldsymbol{x}^* = \boldsymbol{0}$ *of the autonomous system* $\sum$ *is said to be* ***stable*** *according to* ***Lyapunov****, if for every* ϵ*, a* δ *exists such that:*

$$\|\boldsymbol{x_0}\| < \delta\left(\epsilon\right)\ \Rightarrow\ \ \|\boldsymbol{x}\left(t\right)\| < \epsilon\ \forall t > 0\ . \tag{2.8}$$

Definition 2.1.7. *[Ada15] [Lun16]*
The equilibrium $\boldsymbol{x}^*$ *is said to be locally (globally)* ***asymptotically stable*** *if it is stable according to Lyapunov and, furthermore, locally (globally) attractive.*
Or: if $\boldsymbol{x}^*$ *f it is stable according to Lyapunov and, furthermore*

$$\lim_{t\to\infty} \|\boldsymbol{x}\left(t\right)\| = \boldsymbol{x}^* \tag{2.9}$$

holds.

Definition 2.1.8. *[Ada15], [Lun16]*
The function $V\left(\boldsymbol{x}\right) : U\left(\boldsymbol{x}^*\right) \to \mathbb{R}$ *is called a* ***Lyapunov-function*** *for the autonomous system* $\sum$*, if it fulfils the following conditions:*

1. $V\left(\boldsymbol{x}\right)$ *is continuous,* $V\left(\boldsymbol{x} = \boldsymbol{x}^*\right) = 0$ *and* $\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}$ *exists.*
2. $V\left(\boldsymbol{x}\right) > 0$ *for* $\boldsymbol{x} \neq \boldsymbol{x}^*$
3. $\dot{V}\left(\boldsymbol{x}\right) = \dfrac{\partial^T V\left(\boldsymbol{x}\right)}{\partial \boldsymbol{x}} \boldsymbol{f}\left(\boldsymbol{x}\right) \leq 0$

Definition 2.1.9. *The function $V(\boldsymbol{x})$ is called* ***radially unbounded*** *if*

$$\|\boldsymbol{x}\| \to \infty \Rightarrow V(\boldsymbol{x}) \to \infty \tag{2.10}$$

holds.

Following definitions are mainly based on [Sch00] and [Kot10].

Definition 2.1.10.
The ***inner product*** *of two signals $\boldsymbol{f}(t)$ and $\boldsymbol{g}(t)$ is defined by*

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \int_0^\infty \boldsymbol{f}^T \boldsymbol{g} \, dt \ . \tag{2.11}$$

Definition 2.1.11.
The ***$\boldsymbol{L}_2$-norm*** *of a signal $\boldsymbol{f}(t)$ is defined by*

$$\|\boldsymbol{f}\|_2 = \sqrt{\langle \boldsymbol{f}, \boldsymbol{f} \rangle} \ . \tag{2.12}$$

Definition 2.1.12.
A function $v(\boldsymbol{u}, \boldsymbol{y}) : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}$ is called a ***supply rate****.*

Definition 2.1.13.
A function $S(\boldsymbol{x}) : \mathcal{X} \to \mathbb{R}_+$ is called a ***storage function****.*

Definition 2.1.14.
A state space system $\sum$ is said to be ***dissipative*** *with respect to the supply rate v, if there exists a storage function, such that for all $\boldsymbol{x}_0 \in \mathcal{X}$, all $t_1 \geq t_0$, and all input functions $u(t)$*

$$S(\boldsymbol{x}(t_1)) \leq S(\boldsymbol{x}(t_0)) + \int_{t_0}^{t_1} v(\boldsymbol{u}(t), \boldsymbol{y}(t)) dt \tag{2.13}$$

where $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$.

Definition 2.1.14 states that the stored energy in the system at any future time t_1, $S(\boldsymbol{x}(t_1))$, is *at most* equal to the stored energy at the present t_0 plus the externally supplied energy during the time interval $[t_0, t_1]$. In other words, without an external energy supplement, the stored energy in a system cannot increase, or the system can only *dissipate* energy and not create some.

Assuming differentiability of the storage function $S(\boldsymbol{x})$, the *dissipation inequality* (2.1.14) can also be given in differential form as

$$\dot{S}(\boldsymbol{x}(t)) \leq v(\boldsymbol{u}(t), \boldsymbol{y}(t)) \ . \tag{2.14}$$

Definition 2.1.15.
A system $\sum$ *is called* ***lossless****, if in* (2.13) *or* (2.14)*, equality holds.*

Definition 2.1.16. ***Passivity***
A system $\sum$ *is called*

- ***passive*** *if it is dissipative according to Definition 2.1.14 with supply rate*
$$v(\boldsymbol{u}(t), \boldsymbol{y}(t)) = \langle \boldsymbol{y}(t), \boldsymbol{u}(t) \rangle = \boldsymbol{y}(t)^T \cdot \boldsymbol{u}(t) \ . \tag{2.15}$$
- ***strictly input-passive*** *if it is dissipative according to Definition 2.1.14 with supply rate*
$$v(\boldsymbol{u}(t), \boldsymbol{y}(t)) = \langle \boldsymbol{y}(t), \boldsymbol{u}(t) \rangle - \alpha \|u\|^2,\ \alpha > 0 \ . \tag{2.16}$$
- ***strictly output-passive*** *if it is dissipative according to Definition 2.1.14 with the supply rate*
$$v(\boldsymbol{u}(t), \boldsymbol{y}(t)) = \langle \boldsymbol{y}(t), \boldsymbol{u}(t) \rangle - \beta \|y\|^2,\ \beta > 0 \ . \tag{2.17}$$

Definition 2.1.17.
A lossless passive system $\sum$ *is called* ***conservative****.*

The supply rate from (2.15) is the inner product of the input and the output which may be interpreted as *power*. Two well-known examples are mechanical system with generalised forces as input and generalised velocities as output, and, electrical systems with voltages as input and the corresponding currents as output. In this sense, the differential passivity inequality (2.14) states that the increase rate of the energy in a system is bounded by the the power put into it.

Definition 2.1.18.
Suppose a function $f(\boldsymbol{x}) : \mathcal{X} \to \mathbb{R}$*. A point* $\boldsymbol{x}^* \in \mathcal{X}$ *is called a*

- ***local minimum*** *if* $\exists\, \epsilon > 0 : f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}),\ \forall \boldsymbol{x}, \|\boldsymbol{x}\| < \epsilon$*.*
- ***global minimum*** *if* $f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}),\ \forall \boldsymbol{x} \in \mathcal{X}$*.*
- ***global (local) strict minimum*** *if* $<$ *holds instead of* $\leq$*.*

Theorem 2.1.19. ***Stability of dissipative systems*** *[Sch00]*
Suppose a system $\sum$ *with an equilibrium* $\boldsymbol{x}^*$ *is dissipative with regard to the supply rate* v *according to Definition 2.1.14. Suppose further* $\boldsymbol{u} \equiv \boldsymbol{0}$ *and*
$$v(\boldsymbol{0}, \boldsymbol{y}) \leq 0,\ \forall\, \boldsymbol{y}. \tag{2.18}$$
$\boldsymbol{x}^*$ *is locally stable acceding to Lyapunov if it is a local strict minimum of the storage function* $S(\boldsymbol{x})$*. Furthermore,*
$$V(\boldsymbol{x}) = S(\boldsymbol{x}) \,. \tag{2.19}$$
is a Lyapunov-function.

Sine passive systems are an special case of dissipative systems the stability of passive systems is defined based on the stability of the general class, namely dissipative systems. For a passive system, the supply rate is defined as the inner product of the input and the output $\langle \boldsymbol{y}(t), \boldsymbol{u}(t) \rangle$ and, thus, condition (2.18) is directly satisfied. Therefore, the following can be stated:

Theorem 2.1.20. ***Stability of passive systems***
An equilibrium $\boldsymbol{x}^*$ of a passive system is stable according to Lyapunov with the Lyapunov-function $V(\boldsymbol{x}) = S(\boldsymbol{x})$ if it is a strict minimum of $S(\boldsymbol{x})$.

In other words, once a system is passive the strict minimum of its storage function is a stable equilibrium.

Theorem 2.1.21. ***Asymptotic stability of passive system***
Given an input-affine system $\sum_{aff}$ that is both fully reachable and stabilisable. Suppose further that $\sum_{aff}$ is strictly output-passive with the storage function $S(\boldsymbol{x})$ such that

$$S(\boldsymbol{x}) > 0, \forall \boldsymbol{x} \neq \boldsymbol{0}\ , \quad S(\boldsymbol{0}) = \boldsymbol{0}\ . \tag{2.20}$$

Then $\boldsymbol{x}^* = \boldsymbol{0}$ is a local asymptotic stable equilibrium according to Lyapunov.

Definition 2.1.22. ***Generalised Hamiltonian systems*** *[Sch00] [vdSJ14]*
A generalised Hamiltonian system is given by

$$\sum_{GpH} : \begin{cases} \dot{\boldsymbol{x}} = (\boldsymbol{J}(\boldsymbol{x}) - \boldsymbol{R}(\boldsymbol{x})) \frac{\partial^T H(\boldsymbol{x})}{\partial \boldsymbol{x}} + \boldsymbol{G}_{pH}(\boldsymbol{x})\, \boldsymbol{u}, & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{u} \in \mathcal{U} \\ \boldsymbol{y} = \boldsymbol{G}_{pH}^T(\boldsymbol{x}) \frac{\partial^T H(\boldsymbol{x})}{\partial \boldsymbol{x}} & \boldsymbol{y} \in \mathcal{Y}. \end{cases} \tag{2.21}$$

*$H(\boldsymbol{x}) : \mathcal{X} \to \mathbb{R}$ is called the **Hamiltonian** or the Hamiltonian function and the skew-symmetric matrix $\boldsymbol{J}(\boldsymbol{x}) \in \mathbb{R}^{n \times n}$*

$$\boldsymbol{J}(\boldsymbol{x}) = -\boldsymbol{J}^T(\boldsymbol{x}) \tag{2.22}$$

the structure matrix. The symmetric and positive semi-definite matrix $\boldsymbol{R}(\boldsymbol{x}) \in \mathbb{R}^{n \times n}$

$$\boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{R}^T(\boldsymbol{x}) \geq 0. \tag{2.23}$$

is called the damping matrix.

$\sum_{GpH}$ is often called a **port-Hamiltonian** system since energy can be inserted into the system by the *port* containing the input and output $(\boldsymbol{u}, \boldsymbol{y})$. The structure matrix corresponds to the energy exchange in the system and the damping matrix corresponds to the energy dissipation. A Hamiltonian system with $\boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{0}$ is a lossless system.