Introduction

This thesis deals with specific problems arising in the context of signal analysis. In general the main goal in signal analysis is the efficient extraction of information from a given signal. For this purpose the signal—usually modeled in suitable function spaces—needs to be preprocessed, denoised, compressed, etc. There are two main steps necessary for this. The first step is to apply a suitable transformation specifically designed to extract the desired properties, for example the wavelet transform, Gabor transform or shearlet transform. Which transform to choose obviously depends on what type of information one wants to extract. Many interesting transformations are connected to representations of certain locally compact groups. For instance, the wavelet transform is associated with the affine group whereas the Gabor transform is related to the Weyl-Heisenberg group. After transforming, the second step is to decompose the signal into universal building blocks that match the application. Similarly, this process needs to be reversible, thus a reconstruction technique is necessary to regain signals from their discrete decomposition. A stable discretization of signals is inevitable for applications, such as numerics. In order to perform computations involving the signal, a digital representation of the signal is needed. And for this the first step is a uniform discretization of families of signals, that is, the decomposition and reconstruction is performed according to a fixed technique.

Coorbit theory is designed to find function spaces related to transformations of functions that are related to representations of locally compact groups and to describe uniform discretization techniques for these spaces. This theory was originally developed by Feichtinger and Gröchenig [44–46, 67, 68] in the 1980s. The main idea is to measure the smoothness of a function via properties of the transform of the signal. To be more precise, one asks whether the transform is contained in certain function spaces on the index set of the transform, which usually is the underlying group, and this allows to define spaces of functions associated to the transform. Moreover, coorbit theory provides a uniform approach to discretize these function spaces and to characterize the associated discrete sequence spaces, where the sequence spaces directly depend on the group structure. This way the existence of Banach frames and atomic decompositions is naturally ensured. By an application of this theory classical homogeneous Besov-Triebel-Lizorkin spaces [97, 98, 100] can be identified as coorbit spaces [101] via the wavelet transform. Similarly, modulation spaces are related to the Gabor transform [41, 69], Bergman spaces can be treated as well [49] and there are applications of the classical coorbit theory to various shearlet transforms [25].

Classical Coorbit Theory

The original coorbit theory as developed by Feichtinger and Gröchenig [44–46, 67, 68] relies on a locally compact topological group G and a representation π of the group on a Hilbert space $\mathcal{H}$. Associated to an admissible vector $\psi \in \mathcal{H}$ the voice transform V_ψ is given as the map

$$V_\psi \colon \mathcal{H} \to L_2(G), \quad V_\psi f(x) = \langle f, \pi(x)\psi \rangle_{\mathcal{H}}.$$

The admissibility of ψ is just the well-definedness of the map above, we also say the map is square-integrable. This voice transform takes the role of the afore mentioned transform related to the group G and it lays the foundation for the coorbit spaces. One very important property of the voice transform is the fact that the convolution from the right with the element $V_\psi\psi$ is the identity. By denoting with $K_\psi = V_\psi\psi$ the so-called reproducing kernel, then this means $V_\psi f * K_\psi = V_\psi f$ for all $f \in \mathcal{H}$. For some fixed weight w on G we can now define the spaces $\mathcal{H}_{1,w} = \{f \in \mathcal{H} : V_\psi f \in L_{1,w}(G)\}$. The motivation behind these spaces is that we are interested in their dual space $\mathcal{H}'_{1,w}$. This set of distributions will then be the reservoir from which we choose the elements of the coorbit spaces. In order to do so we first extend the voice transform to the dual space $\mathcal{H}'_{1,w}$ via a Gelfand triple setting and denote the extension with $V_{e,\psi}$. Then, we are interested in those functionals in $\mathcal{H}'_{1,w}$, whose extended voice transform decays rapidly enough. In other words, they need to be contained in a certain function space Y, which characterizes the decay of transforms:

$$\mathrm{Co}(Y) = \{T \in \mathcal{H}'_{1,w} : V_{e,\psi}T \in Y\}.$$

Examples for the space Y include L_p-spaces and their weighted counterparts. Under certain assumptions these coorbit spaces are well-defined Banach spaces. Another very important and much used property of the coorbit spaces is that they are isometrically isomorphic to the reproducing kernel Banach space $\{f \in Y : f * K_\psi = f\}$. This means it suffices to prove almost all properties for the latter space and then transfer them to the coorbit space. In other words, the reproducing kernel space already contains all structural properties. Finally, the coorbit theory provides a unified approach to discretize coorbit spaces. To be more precise, the existence of atomic decompositions and Banach frames is ensured. This is a very powerful tool to analyze the structure and other properties of coorbit spaces.

Coorbit Theory with Non-Integrable Kernel

The theory above is based on several fundamental assumptions, one of them states that the representation π is not only square-integrable but also integrable. This means the reproducing kernel K_ψ is an element of $L_{1,w}(G)$. Unfortunately, this condition is quite restrictive and even in the example of Paley-Wiener spaces not fulfilled. To work around this Dahlke et al. [18] have developed a coorbit theory for those spaces where the kernel is contained in a general Fréchet space of functions. This includes the setting where K_ψ is not contained in $L_{1,w}(G)$ but in all weighted Lebesgue spaces $L_{p,w}(G)$ for $1 < p < \infty$. We also say the kernel is non-integrable. In fact, it has been shown that also in this setting associated coorbit spaces are well-defined; the theory, however, needs certain adjustments. We need, for example, a substitute for the space $\mathcal{H}_{1,w}$ and accordingly the voice transform needs to be extended differently.

Yet, one question was left unanswered in [18]. It was unclear how one can obtain discretization results for these spaces as in the classical case, including atomic decompositions and Banach frames. The answer to this question is a part of this thesis and the results have been published in [19]. The first surprising result is the fact that the methods from the classical setting do not carry over to the Fréchet setting. Instead we present a different result that does not require the integrability of the kernel and therefore uses convolution inequalities other than Young's inequality. More precisely, Theorem 3.2.17 includes the following statement concerning reconstruction operators: for sequences $d \in \ell_{q,m}(I)$ with countable index set I in certain weighted sequence spaces we have

$$T = \sum_{i \in I} d_i \pi(x_i)\psi \in \mathrm{Co}(L_{r,m}) \quad \text{for } q < r.$$

Here, $(x_i)_{i\in I}$ are certain points in G that are sufficiently close together. The price we pay for this result is that the integrability parameters of the discrete norm of the coefficient space and of the coorbit norm are different. This may appear strange at first but is a direct consequence of the altered convolution inequalities we employ. Conversely, we also obtain decompositions of functions, but only approximative. This means for every $\varepsilon > 0$ there is a finite sequence $c \in \ell_{r,m}(J)$ such that

$$\Big\| T - \sum_{j\in J} c_j \pi(x_j)\psi \Big\|_{\mathrm{Co}(L_{r,m})} \leqslant \varepsilon.$$

Both the reconstruction and the decomposition of functions is continuous, but not uniform.

The decomposition techniques described above are based on one assumption, namely we assume the convolution with the reproducing kernel K_ψ is a bounded operator on $L_{r,m}(G)$. This is not clear for non-integrable kernel and is therefore an additional condition. In Theorem 3.3.1 we show, however, that this condition is not only sufficient for a first discretization but also necessary for atomic decompositions and Banach frames for coorbit spaces. Therefore the assumption is inevitable and it seems like we have reached the limit of what is possible.

Still, the results above are not optimal. To obtain proper atomic decompositions and Banach frames similar to the classical coorbit theory we introduce the additional assumption that there is a second kernel W on G satisfying, among others, the following conditions:

(i) $W \in L_{1,w}(G)$,

(ii) $W * K_\psi = K_\psi$ or $K_\psi * W = K_\psi$, respectively.

With these properties at hand, Theorem 3.4.8 shows that the family $(\pi(x_i)\psi)_{i\in I}$ constitutes a Banach frame for suitably chosen discrete points $(x_i)_{i\in I} \subset G$. Analogously, Theorem 3.4.15 shows that under very similar conditions the same family of functions paves the way for atomic decompositions. It is therefore possible to describe exact conditions under which the same discretization results hold for coorbit spaces with non-integrable kernel as in the classical setting.

Generalized Coorbit Theory

The extension of coorbit theory described above is not the only generalization developed. In the last 15 years the coorbit theory has been rediscovered and generalized in many forms. For example, the classical theory only admitted Banach spaces of functions Y as target spaces and this was later extended to quasi-Banach spaces [90]. In [79] this has been further expanded to the setting of quasi-Banach spaces with variable smoothness and integrability. Moreover it was recognized that many interesting examples do not fit the group setting described above and the theory was generalized by Dahlke et al. to also fit the setting of homogeneous spaces, that is, quotients of groups via subgroups G/H [21, 26, 27]. This allowed to view modulation spaces on spheres from the standpoint of coorbit theory, as well as α-modulation spaces. The latter spaces were originally introduced by Feichtinger and Gröbner and can be seen as a merging of modulation spaces and Besov spaces [43]. Another approach is to exploit symmetry properties of functions, where a group is considered modulo a symmetry group, like radial symmetry [88, 89].

The approach we will discuss in the following is based on the realization that group theory is not needed at all to develop a coorbit theory [53]. This was originally proposed by Fornasier and Rauhut and later extended [76, 102]. The main idea is, instead of having a group, to take an arbitrary measure space (X, μ) as the index set of our transform. Without a group structure available the notion of representations of said groups on Hilbert spaces makes no sense. Instead, we replace both by continuous frames [1]. Assume we have a family of functions $\mathfrak{F} = (\psi_x)_{x\in X}$

indexed by the measure space X, which constitutes a continuous frame for the Hilbert space $\mathcal{H}$. Then the frame elements replace the functions $\pi(x)\psi$ above and therefore the voice transform associated to $\mathfrak{F}$ is defined via

$$V_{\mathfrak{F}} \colon \mathcal{H} \to L_2(X, \mu), \quad V_{\mathfrak{F}} f(x) = \langle f, \psi_x \rangle_{\mathcal{H}}.$$

Contrary to the setting before, we have no convolution available on arbitrary measure spaces. Instead, we assign a kernel operator to functions $K \colon X \times X \to \mathbb{C}$ via the integral

$$Kf(x) = \int_X K(x, y) f(y)\, d\mu(y).$$

If we now look at the kernel operator associated to the kernel $K_{\mathfrak{F}}(x, y) = V_{\mathfrak{F}}\psi_y(x)$, then the operator fulfills a similar reproducing identity as above given by $K_{\mathfrak{F}} V_{\mathfrak{F}} f = V_{\mathfrak{F}} f$ for all $f \in \mathcal{H}$. In this sense the kernel operator is a substitute, or generalization, of convolution. To measure the integrability of the reproducing kernel we introduce weighted kernel spaces $\mathcal{A}_{p,w}$ for weights w on $X \times X$ and integrability parameters $1 \leqslant p \leqslant \infty$. Obviously, we need substitutes for Young's inequality for the kernel spaces, which we provide. Then, by assuming $K_{\mathfrak{F}} \in \mathcal{A}_{1,w}$, in a certain sense the integrability of the kernel, we can define the test spaces $\mathcal{H}_{1,v} = \{f \in \mathcal{H} : V_{\mathfrak{F}} f \in L_{1,v}\}$ similar to the classical case. Again, the voice transform can be extended by $V_{e,\mathfrak{F}}$ to the dual space $\mathcal{H}'_{1,v}$, which serves as a reservoir for the coorbit spaces. For suitable function spaces Y measuring the decay rate of the voice transform, the corresponding coorbit spaces are given by

$$\mathrm{Co}(Y) = \{T \in \mathcal{H}'_{1,v} : V_{e,\mathfrak{F}} T \in Y\}.$$

These spaces have the same properties as above, that is, they are Banach spaces and they are isometrically isomorphic to the reproducing kernel space $\{f \in Y : K_{\mathfrak{F}} f = f\}$. Likewise, we can pose conditions under which the existence of both atomic decompositions and Banach frames is ensured.

Generalized Coorbit Theory with Non-Integrable Kernel

As in the group setting, the theory described above relies on several fundamental assumptions. One assumption is named above, namely the integrability of the reproducing kernel with respect to weighted kernel spaces: $K_{\mathfrak{F}} \in \mathcal{A}_{1,w}$. This assumption is restrictive and not always fulfilled. We therefore develop a new theory that allows frames indexed by arbitrary measure spaces but also includes $K_{\mathfrak{F}} \notin \mathcal{A}_{1,w}$. Parts of the results have been published in [50]. The main idea is to assume $K_{\mathfrak{F}} \in \mathcal{A}_{p,w}$ for all $1 < p < \infty$, which is a weaker assumption than the integrability above. This setting is similar to coorbit theory with non-integrable kernel and group structure. Also in this case we can find substitutes for the test space $\mathcal{H}_{1,v}$ and it is possible to properly define meaningful coorbit spaces.

Yet again we are faced with the challenge of finding discretizations for these new coorbit spaces. For this we apply similar ideas as the ones for coorbit spaces with non-integrable kernel in the group case and obtain comparable results. To be more precise, we show the following reconstruction result in Theorem 5.2.17: for a countable sequence $d \in \ell_{q,m}(I)$ in a certain weighted sequence space we have

$$T = \sum_{i \in I} d_i \psi_{x_i} \in \mathrm{Co}(L_{r,m}) \quad \text{for } q < r,$$

where the discrete points $(x_i)_{i \in I} \subset X$ are chosen sufficiently close together, which is achieved using suitable coverings of the index set. Again, the downside is the different integrability parameters of the discrete norm of the coefficient space and the coorbit norm. Conversely, we can decompose the elements approximately in the following fashion. Take $\varepsilon > 0$, then there is a finite sequence $c \in \ell_{r,m}(J)$ such that

$$\Big\| T - \sum_{j\in J} c_j \psi_{x_j} \Big\|_{\mathrm{Co}(L_{r,m})} \leqslant \varepsilon.$$

These results require an additional assumption on the reproducing kernel $K_{\mathfrak{F}}$, that is, we assume the corresponding kernel operator is bounded as an operator on weighted Lebesgue spaces $L_{p,m}(X,\mu)$. This assumption needs to be checked individually for applications. Furthermore, we show in Theorem 5.3.1 that this assumption is also necessary for atomic decompositions and Banach frames to exist. So again, the boundedness of the kernel operator is inevitable and it seems like we have reached the limit of what is possible here as well.

As before, however, the results can still be improved under the following assumption. We assume there exists a second kernel W which, among others, fulfills the following conditions:

(i) $W \in \mathcal{A}_{1,w}$,

(ii) $W \circ K_{\mathfrak{F}} = K_{\mathfrak{F}}$ or $K_{\mathfrak{F}} \circ W = K_{\mathfrak{F}}$, respectively.

The operation $\circ$ denotes the multiplication of two kernels. This additional kernel W allows us to prove in Theorem 5.4.7 that the family $(\psi_{x_i})_{i\in I}$ constitutes a Banach frame for $\mathrm{Co}(L_{r,m})$ for suitably chosen points $(x_i)_{i\in I}$. And similarly Theorem 5.4.11 shows that under very similar assumptions and using the same family of functions the existence of atomic decompositions is also ensured.

Applications

There are numerous applications of coorbit theory, some were mentioned above. The following two additional applications provide good examples of the developed theory. The Paley-Wiener spaces illustrate how the coorbit theory for non-integrable kernels in the group setting can be applied to find proper discretization results. And we apply the ideas of generalized coorbit spaces with non-integrable kernel to certain shearlet frames to define new function spaces.

Paley-Wiener Spaces

The Paley-Wiener space B^p_Ω is the collection of functions in the Lebesgue space $L_p(\mathbb{R})$, where the Fourier transform is supported in a fixed subset $\Omega \subset \mathbb{R}$. In [18] it was shown that these spaces can be interpreted as coorbit spaces with non-integrable kernel and the underlying group is the additive group $\mathbb{R}$. The reproducing kernel is then given by the sinc-function, which is clearly not an element of $L_1(\mathbb{R})$. The developed theory is therefore applicable and we show that for different subsets Ω, the Paley-Wiener spaces represent both positive and negative examples for the discretization ideas. The negative examples include compact subsets, for which the convolution operator associated with the reproducing kernel is not bounded on Lebesgue spaces. Hence, the discretization techniques are not applicable. There are, however, also positive examples where the contrary is true and we can indeed show the existence of Banach frames and atomic decompositions. This includes the symmetric intervals $\Omega = [-\omega, \omega]$ and provides an alternative proof for the Whittaker-Kotelnikov-Shannon sampling theorem [63, 96].

Shearlets

As mentioned above, the classical coorbit theory can be applied to wavelets, yielding homogeneous Besov spaces. While wavelets are especially suitable in signal analysis for identifying isolated singularities, they are less efficient when dealing with signals with anisotropic features due to their isotropic nature. Since the identification of anisotropic features of signals, such as directional information, is of great importance in practice, other directional representation systems have been developed like curvelets [10, 12, 94], ridgelets [11], contourlets [34, 35] or shearlets.

Shearlets were designed as an extension to wavelets in multiple dimensions. While wavelet systems only consist of isotropically dilated and translated versions of a mother function, shearlet systems consist of anisotropically dilated, translated and sheared copies of the function. The additional shearing parameter allows changing the direction of the function, which is supported by the anisotropy. This makes them especially well-suited to deal with localized directional features in a signal. Indeed, it was shown in [80] that the shearlet transform can be used to resolve the wavefront set of a signal and in [72] that the shearlets yield an optimal N-term approximation error for cartoon-like functions. For further insights on shearlet algorithms we refer to [81].

Another great advantage of shearlets, which sets them apart from the other systems mentioned above, is the fact that the continuous shearlet transform, introduced and investigated in [24, 25, 28, 71], stems from the action of a square-integrable representation of a topological group, the so-called full shearlet group $\mathbb{S} = \mathbb{R}^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d$. This makes it possible to apply the classical coorbit theory described above, which was investigated by Dahlke et al. in a series of papers [22, 29]. Since the shearlets used in the construction of these shearlet coorbit spaces are required to have vanishing moments, any polynomial part in a signal is ignored by the transform. This leads to the resulting spaces being homogeneous spaces, in the sense that, intuitively speaking, the shearlet transform possesses a “blind spot” in the Fourier domain. In practice, however, the smoothness spaces being used, for example to analyze the regularity of the solution space of an operator equation, are not homogeneous. This calls for inhomogeneous smoothness spaces related to the shearlet transform. We note there are already approaches, though not based on coorbit theory, to develop inhmogeneous shearlet smoothness spaces. In [82] Labate et al. used the notion of decomposition spaces [42,43], while in [103,104] Vera applied the framework of the φ-transform, introduced by Frazier and Jawerth [54], for this purpose.

In our approach we use the ideas of the inhomogeneous wavelet transform [53] and transport them to the shearlet group. This means we define the index space

$$X = \big(\{\infty\} \times \mathbb{R}^{d-1} \times \mathbb{R}^d\big) \cup \big([-1,1]\backslash\{0\} \times \mathbb{R}^{d-1} \times \mathbb{R}^d\big),$$

where the left part is designed to analyze the low frequency part of a signal, which is ignored in the homogeneous setting. Using this index space we define a corresponding frame $\mathfrak{F} = (\psi_x)_{x\in X}$ via

$$\psi_{(\infty,s,t)} = \Phi(S_s^{-1}(\cdot - t)) \quad \text{and} \quad \psi_{(a,s,t)} = |\det A_a|^{-1/2}\Psi(A_a^{-1}S_s^{-1}(\cdot - t)),$$

where the functions Φ and Ψ can be chosen such that $\mathfrak{F}$ is a continuous frame for $L_2(\mathbb{R}^d)$, the so-called inhomogeneous shearlet frame. The corresponding reproducing kernel is in fact contained in all weighted kernel space $\mathcal{A}_{p,w}$ for $p > 1$ such that the generalized coorbit theory with non-integrable kernel is applicable. This way we can introduce new inhomogeneous shearlet coorbit spaces and give a first reconstruction result for these spaces.

Outline

This thesis starts in Chapter 1 with some preliminaries. This includes standard concepts as well as new results, including new convolution inequalities for weighted Lebesgue spaces similar to Young’s inequality in Subsection 1.3.2 and new weighted kernel spaces for arbitrary measure spaces including norm inequalities for kernel operators in Section 1.8. In Chapter 2 we recall the classical coorbit theory and their discretization techniques, as well as two important applications of the theory, namely homogeneous Besov spaces and shearlet coorbit spaces. Then, in Chapter 3 we recall the coorbit theory with non-integrable kernel and show new discretization results. We add several applications of the Paley-Wiener spaces to motivate the theory. In Chapter 4 we recall the generalized coorbit theory as well as their discretizations and include the example of inhomogeneous Besov spaces. Finally, in Chapter 5 we present the new generalized coorbit theory with non-integrable kernel and prove the well-definedness of coorbit spaces and some discretization results. This is followed by the application to the newly defined inhomogeneous shearlet coorbit spaces. In the final Chapter 6 we include some conclusions and discuss proceeding problems and ideas that might invigorate further research.

Chapter 1

Preliminaries

The aim of this chapter is to recall and to introduce some mathematical concepts and notations forming the foundation of this dissertation. While some of these concepts are standard knowledge in applied and harmonic analysis, others are not very common and in parts altered or complemented for our setting.

We start by introducing notational conventions in Section 1.1 followed by brief insights in well-known areas of group theory, function space theory and representation theory for groups in Sections 1.2, 1.3 and 1.4, respectively. In Section 1.5 we introduce coverings of index spaces, which are used in Section 1.6 to define sequence spaces associated with certain spaces of functions. Then, in Section 1.7 we recall the theory of continuous frames followed by a definition of kernel spaces and a collection of important properties in Section 1.8. Finally, in Section 1.9 we recall two fundamental discretization techniques for function spaces, namely atomic decompositions and Banach frames.

1.1 Notations and Conventions

Here, we fix standard notations used throughout this thesis. More specific notations are introduced in the text and not listed here.

- The letters $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ have their usual meaning. We denote the natural numbers by $\mathbb{N} = \{n \in \mathbb{Z} : n \geqslant 1\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We also use the conventions $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and $\mathbb{R}_+$ for all positive real numbers and $\mathbb{R}_{\geqslant 0}$ for all non-negative real numbers.

- For a dimension $d \in \mathbb{N}$ we denote with $\mathbb{R}^d$ the Euclidean d-space and for two elements $x, y \in \mathbb{R}^d$ we use the canonical inner product

  $$\langle x, y\rangle = \sum_{i=1}^{d} x_i y_i.$$

  Any other inner product will be provided with a subscript.

- If $d \geqslant 2$ we write $x = (x_1, \widetilde{x})$, where $\widetilde{x} = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$

- We write $I_d \in \mathbb{R}^{d\times d}$ for the d-dimensional identity matrix and $0_d \in \mathbb{R}^d$ for a vector containing only zeros.

- For $x \in \mathbb{R}^d$ we use $\|x\|_2$ for the Euclidean norm, $\|x\|_1$ for the 1-norm and $\|x\|_\infty$ for the maximum norm of x.

- If $A \in \mathbb{R}^{d\times d}$ is a matrix, $\|A\|_{2\to 2}$ denotes the spectral norm of A.

- For two sets X, Y we write $X \subset Y$ if X is a proper subset of Y with $X \neq Y$ and we write $X \subseteq Y$ if we allow $X = Y$.

- For a set X and any subset $M \subset X$ the indicator function $\chi_M : X \to \{0,1\}$ is defined as

$$\chi_M(x) = \begin{cases} 1, & \text{if } x \in M, \\ 0, & \text{if } x \notin M. \end{cases}$$

- If X is a topological space and $M \subset X$ a subset, we denote with $\overline{M}$ the closure of M and with $\mathring{M} = \operatorname{int} M$ the interior of M.

- The Lebesgue measure on $\mathbb{R}^d$ is denoted by $\mathrm{d}x$.

- For a function $f : \mathbb{R} \to \mathbb{C}$ and $n \in \mathbb{N}$ the n-th derivative is denoted by $\frac{\mathrm{d}^n f}{\mathrm{d}x^n}$.

- For a function $g : \mathbb{R}^d \to \mathbb{C}$ and a multiindex $\alpha \in \mathbb{N}_0^d$ the partial derivative is denoted by $\frac{\partial^\alpha g}{\partial x^\alpha}$. If $\alpha \in \mathbb{N}_0^d$ is the i-th unit vector then the i-th partial derivative is given by $\frac{\partial g}{\partial x_i}$.

- We denote with $\mathcal{C}^k(\mathbb{R}^d)$, $k \in \mathbb{N}_0$, the set of functions $f : \mathbb{R}^d \to \mathbb{C}$ for which all partial derivatives $\frac{\partial^\alpha f}{\partial x^\alpha}$, $|\alpha| = \alpha_1 + \ldots + \alpha_d \leqslant k$, exist and are continuous.

- The space $\mathcal{C}_0^\infty(\mathbb{R}^d)$ denotes the space of smooth functions on $\mathbb{R}^d$ with compact support.

- We use $\mathcal{S}(\mathbb{R}^d)$ for the space of Schwartz functions on $\mathbb{R}^d$.

- If X is a topological space then $\mathcal{C}_c(X)$ denotes the space of compactly supported and continuous functions on X and $L_0(X)$ denotes the space of all equivalence classes of measurable functions on X.

- We write L_p, $0 < p \leqslant \infty$, for the Lebesgue spaces on a suitable measure space. For an integrability parameter $1 \leqslant p \leqslant \infty$ we denote with $p' = \frac{p}{p-1}$ the Hölder-dual of p.

- For topological spaces X, Y we write $X \hookrightarrow Y$ for the embedding of X in Y, i.e., $X \subseteq Y$ and the identity map is continuous.

- Concerning the Fourier transform of a function $f \in L_1(\mathbb{R}^d)$ we write $\widehat{f} = \mathcal{F}f$ using the convention

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \omega, x \rangle} \,\mathrm{d}x.$$

  The inverse Fourier transform is likewise given by

$$\mathcal{F}^{-1}f(x) = \int_{\mathbb{R}^d} f(\omega) e^{2\pi i \langle \omega, x \rangle} \,\mathrm{d}x.$$

  We use the same symbols for the unitary automorphism on $L_2(\mathbb{R}^d)$.

- For quantities a and b we write $a \lesssim b$ if there exists a finite constant $C > 0$ such that $a \leqslant C \cdot b$, with C being independent of all relevant parameters. If $a \lesssim b$ and $b \lesssim a$ we write $a \asymp b$.