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Lower dimensional models in elasticity

With the purpose of fixing notation and nomenclature, we begin by quickly reviewing some fundamental notions in elasticity theory.[1.1] We then discuss dimension reduction in this context and its mathematical justification. We continue with a brief review of the literature where Γ-convergence is applied for this purpose, to conclude with an outline of the present work and some acknowledgements. Please refer to Appendix B for the notation used throughout this work.

1.1 Elasticity, in a rush

The objects of study are a three dimensional **body** identified with an open, bounded and Lipschitz set $\Omega \subset \mathbb{R}^3$ and its **deformation** $y \colon \Omega \rightarrow \mathbb{R}^3$ under external forces or boundary conditions. When deformations can be assumed to be very small it is more convenient to use instead **displacements** $w \colon \Omega \rightarrow \mathbb{R}^3$, defined by $y(x) = x + w(x)$. Throughout we employ so-called **Lagrangian coordinates**, i.e. we track the deformations of material points wrt. the fixed domain Ω.[1.2]

Subject to external forces or boundary conditions, bodies deform. The fundamental assumption is that any deformation which is not a **rigid body motion** (the composition of a translation and a rotation) stores **elastic**

1.1. A thorough introduction to elasticity can be found in [Cia88], a gentle one from the perspective of differential geometry in [Cia05] and a deeper one in [MH94]. For a very good exposition of continuum mechanics with elasticity as an application see [TM05].

1.2. As opposed to the Eulerian description which instead tracks locations in space.

energy into the body which can be released after the extraneous conditions disappear and this release will bring the body back to its **reference configuration** Ω, without inducing any permanent alteration. If this does not hold, that is, in case the properties of the body are changed after the forces disappear, one can have **viscoelastic** or **plastic** behaviour, but we will not concern ourselves with these at all. If the reference configuration has zero elastic energy, we speak of a **natural state**. The elastic energy can be computed as the integral over Ω of a **stored energy density** W, which under mild assumptions turns out to be a function only of the position $x \in \Omega$ and the **deformation gradient** $\nabla y(x)$. When this is the case we speak of a **hyperelastic** material. The function W expresses the relationship between **strains** (local elongations and compressions in each direction) and **stresses** (internal forces induced by the strains). By our fundamental assumption above, W is non-negative and vanishes for rigid motions, or $W(x, \nabla y) = 0$ for all $\nabla y \in \mathrm{SO}(3)$.

We model the strain by the change in metric induced by the map y in the body wrt. the flat metric, via the so-called **Green - St.Venant's tensor** $E(y) = \frac{1}{2}(\nabla^\top y\, \nabla y - I)$. In terms of displacements $w = y - \mathrm{id}$, this is $E(w) = \frac{1}{2}(\nabla^\top w + \nabla w + \nabla^\top w\, \nabla w)$. Now we can characterise a **rigid motion** or **rigid body movement** as a deformation y such that $E(y) = 0$, i.e. $\nabla^\top y\, \nabla y = I$, since there is no change in the distance between deformed points. The set of all rigid motions consists of all maps $x \mapsto Q\,x + c$ with $Q \in \mathrm{SO}(3)$, $c \in \mathbb{R}^3$. Under the assumptions that displacements are “infinitesimally smaller” than the characteristic dimensions of the body, E is approximated by the **linear strain tensor** $e(w) := \nabla_s w = (\nabla^\top w + \nabla w)/2$ and one speaks of **geometrically linear** elasticity.

Assuming a smooth energy density and a small displacement gradient $\|\nabla w\| \ll 1$, one can linearise the energy around the identity:

$$
\begin{aligned}
W(\nabla y) &= W(I) + DW(I)[\nabla w] + \tfrac{1}{2} D^2 W(I)[\nabla w, \nabla w] + h.o.t. \\
&\approx \tfrac{1}{2} D^2 W(I)[\nabla w, \nabla w] \\
&=: \tfrac{1}{2} Q_3(\nabla w),
\end{aligned}
$$

where we used that W vanishes on rigid motions so, in particular $W(I)$ and $DW(I)$ are zero, and where Q_3 is the **quadratic form of linear elas-**

ticity. In this setting we speak of **linearly elastic** materials. The form Q_3 vanishes exactly over the set of **linearised rigid motions**[1.3]

$$\mathscr{R} := \{x \mapsto Rx + b : R \in \mathrm{so}(3), b \in \mathbb{R}^3\} = \{x \mapsto r \times x + b : r, b \in \mathbb{R}^3\},$$

where so(3) is the space of antisymmetric matrices.

In order to define Q_3 in terms of the gradients ∇w one needs so-called **constitutive relations** between stresses and strains, which may take into account properties like **isotropy** (the body exhibits no "preferred direction" along which responses are different) and **homogeneity** (the body has the same behaviour at any material point $x \in \Omega$). The symmetries arising in isotropic, homogeneous materials imply that Q_3 has the form

$$Q_3(F) = \lambda \operatorname{tr}^2 F + 2\mu |F|^2$$

where $F = \nabla w \in \mathbb{R}^{3 \times 3}_{\mathrm{sym}}$ is a strain tensor and λ, μ are the **Lamé constants** of the material.

There are several other couples of physically meaningful magnitudes related to these two constants, among which we mention **Young's modulus** E and **Poisson's ratio** ν since we use them in the implementation of the discretisations. E is a measure of how the body extends or contracts in response to tensile or compressive stresses. ν measures the tendency of materials to compress in directions perpendicular to the direction of elongation.[1.4]

1.3. In the setting of very small displacements, one must exclude symmetries (large displacements) from rigid motions, which means that the rotation matrices Q do not have the eigenvalue -1 and the maps $I + Q$ are invertible. Then we can define $R := (I - Q)(I + Q)^{-1}$ and recover Q with **Cayley's transform** $R \mapsto (I - R)(I + R)^{-1} = Q$. This bijection allows the identification of matrices Q with matrices R, so we can focus on maps $x \mapsto Rx + b$ with $R \in \mathrm{so}(3)$. Additionally, each R is determined by just 3 coefficients, so there exists a vector $r \in \mathbb{R}^3$ such that $Rx + b = r \times x + b$.

1.4. E is defined as the quotients of stresses over strains along each direction, which reduces to a number for isotropic materials. Since strains are dimensionless, it has units of pressure N/m^2 or Pa, with typical values in the mega- and gigapascal range. ν is the quotient of transverse strain to axial strain, with a sign, for each direction. Again, for isotropic materials this is only a number. Typical values range from 0 for materials with insignificant transversal expansion when compressed (e.g. cork) to 0.5 for incompressible ones (e.g. rubber), but materials have been designed beyond this range (*auxetic metamaterials*).